GROUP ACTION ON (M, N)-FUZZY GROUP

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Abstract

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1 Introduction

Satya Saibaba [2008] initiate the study of L-fuzzy lattice ordered groups and introducing the notice of L-fuzzy sub l-groups. Goguen [1967] replaced the valuation set [0, 1] by means of a complete lattice in an attempt to make a generalized study of fuzzy set theory by studying L-fuzzy sets. Solairaju and Nagarajan [2009] introduced the concept of lattice valued Q-fuzzy submodules over near rings with respect to T-norms. Gu [1994] introduced concept of fuzzy groups with operator.

Cho and Jun [2005] discussed the notion of the intuitionistic fuzzification of a right (resp. left) R-subgroup in a near-ring. Kim and Jun [2000] investigated fuzzy algebraic properties in fuzzy R-subgroup in a near-ring.

They [2001] studied on normal fuzzy R-subgroup, and its homomorphic image and pre-images in a near-ring. Kim and Jun [2001] further extended their contributions in anti-fuzzy R-subgroup in a near-ring. They [2002] again analyzed union and intersection of fuzzy R-subgroups in a near-ring.

Solairaju and Nagarajan [2008] initiated some contributions on Q-fuzzy left R-subgroup of near-ring under triangular norm. They [2011] introduced the new structures of Q-fuzzy groups and then they investigate the notion upper Q-fuzzy index order with upper Q-fuzzy subgroups. Osman Kazanci et. al. [2007] discussed the notion of intuitionistic Q-fuzzy R-subgroup in a near ring, and related properties are investigated.

2 Preliminaries

Definition 2.1. *A fuzzy subset on X is a map from X into the closed interval [0, 1].*

Definition 2.2. *A L-fuzzy subset on X is a map from X into a lattice L with the least element 0 and the greatest element 1.*

Definition 2.3. *Let A be a fuzzy subset of X. Them $A(a) = \{x \in X : A(x) \leq a\}$ where $a \in [0, 1]$.*

Definition 2.4. *Let $(S, +)$ be a group, and G be a non-empty set. Then S acts on G if there exists a function $* : S \times G \to G$ (denoted $*(g, s) = gs$ for all $g \in G$, and $s \in S$) such that $es = s$ and $(s + t) * g = s * (t * g)$ for all s, t in S, and for all g in G.*

Notations: Let a, b be elements in as lattice. Throughout this paper, $\max\{a, b\} = a \vee b =$ the least upper bound of a, b in L, and $\min\{a, b\} = a \wedge b$ = the greatest lower bound of a, b in L.

Definition 2.5. *Let a non-empty set K acts on an L-fuzzy set A under a group $(S, .)$. Let m, n be elements in L. Then A is (m, n)-L-fuzzy subgroup acted by K if*

(1) $A(k * (xy)) \vee m \geq (A(k * x) \wedge A(k * y)) \wedge n$

(2) $A(k * x^{-1}) \vee m \geq A(k * x) \wedge n$ *for all* $k \in K$, *and* $x, y \in S$.

3 Contributions on (m, n)-fuzzy group acted by a set

Theorem 3.1. *Let A be a (m, n)-L-fuzzy subgroup acted by a non-empty K, and 1 be the identity in S. Then $A(k*1) \vee m \geq A(k*x) \wedge n$ for all x in S.*

Proof. Let $x \in S$. Then x^{-1} is the inverse of x in S.
It implies that

$$\begin{aligned}
A(k*1) \vee m &= (A(k*xx^{-1}) \vee m) \vee m \\
&\geq (A(k*x) \wedge A(k*x^{-1})) \wedge n) \vee m \\
&\geq (A(k*x) \wedge A(k*x^{-1})) \vee m) \wedge (n \vee m) \\
&\geq (A(k*x) \vee m)) \wedge (A(k*x^{-1}) \vee m)) \wedge (n \vee m) \\
&\geq (A(k*x) \vee m) \wedge (A(k*x) \wedge n)) \wedge (n \vee m) \\
&\geq (A(k*x) \vee m)) \wedge A(k*x) \wedge n \wedge (n \vee m) \\
&\geq (A(k*x) \wedge n \wedge (n \vee m) \\
&\geq (A(k*x) \wedge n.
\end{aligned}$$

□

Theorem 3.2. *Let a non-empty set Kacts on an L-fuzzy set A under a group $(S, .)$. Then A is (m, n)-L-fuzzy subgroup acted by K if and only if $A(k*x^{-1}y) \vee m \geq A(k*x) \wedge A(k*y) \wedge n$ for all x, y in S.*

Proof. Let A is a (m, n)-L-fuzzy subgroup acted by a non-empty set K.
Then

$$\begin{aligned}
A(k*x^{-1}y) \vee m &= (A(k*x^{-1}y) \vee m) \vee m \\
&\geq [A(k*x^{-1}) \wedge A(k*y) \wedge n] \vee m \\
&\geq [(A(k*x^{-1}) \wedge A(k*y)) \vee m] \wedge (n \vee m) \\
&\geq ([A(k*x^{-1}) \vee m] \wedge [A(k*y) \vee m]) \wedge (n \vee m) \\
&\geq ([A(k*x) \wedge n] \wedge A(k*y) \vee (n \vee m) \\
&\geq A(k*x) \wedge n \wedge A(k*y) \wedge (n \vee m) \\
&\geq A(k*x) \wedge A(k*y) \wedge n \wedge (n \vee m) \\
&\geq A(k*x) \wedge A(k*y) \wedge n.
\end{aligned}$$

3

Conversely, $A(K * x^{-1}y) \vee m \geq A(k * x) \wedge A(k * y) \wedge n$ for all x, y in S. Thus

$$A(k * 1) \wedge m \geq A(k * y) \wedge n \text{ for all } y \text{ in } S.$$

$$\begin{aligned} A(k * x^{-1}) \vee m &= (A(k * x^{-1} * 1) \vee m) \vee m \\ &\geq (A(k * x^{-1}y) \wedge n) \vee m \\ &\geq (A(k * x^{-1}y) \vee m) \wedge (n \vee m) \\ &\geq (A(k * x) \wedge A(k * y) \wedge n) \wedge (n \vee m) \\ &\geq A(k * x) \wedge A(k * y) \wedge n. \end{aligned}$$

Now

$$\begin{aligned} A(k * xy) \vee m &\geq A(k * xy) \vee m \vee m \\ &\geq (A(k * x^{-1}) \wedge A(k * y) \wedge n) \vee m \\ &\geq (A(k * x^{-1}) \vee m) \wedge (A(k * y) \wedge n) \vee m) \\ &\geq [A(k * x) \wedge A(k * y) \wedge n] \wedge (A(k * y) \wedge n) \vee m) \\ &\geq [A(k * x) \wedge A(k * y) \wedge n] \wedge (A(k * y) \wedge n) \\ &\geq [A(k * x) \wedge A(k * y) \wedge n]. \end{aligned}$$

In addition,

$$\begin{aligned} A(k * x^{-1}) \vee m &\geq A(k * x^{-1}1) \vee m \vee m \\ &\geq [A(k * x) \wedge A(k * 1) \wedge n] \vee m \text{ for all } x, y \text{ in } S. \\ &\geq ((A(k * x) \wedge n) \vee m) \wedge (A(k * 1) \vee m) \\ &\geq (A(k * x) \wedge n) \wedge (A(k * 1) \vee m) \\ &\geq (A(k * x) \wedge n) \wedge (A(k * x) \wedge n) \\ &\geq (A(k * x) \wedge n). \end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let a non-empty set K acts on an L-fuzzy set A under a group $(S, .)$. Then A is (m, n)-L-fuzzy subgroup acted by K if and only if $A(\alpha)$ is an L-subgroup acted by K for any $\alpha \in [m, n]$ and $A(\alpha) \neq \{\ \}$.*

Proof. Case (1): (i) implies (ii).

Let A be (m,n)-L-fuzzy subgroup acted by K with $A(\alpha) \neq \{\ \}$. Let x, y be in $A(\alpha)$. Then $A(k*x) \geq \alpha$, and $A(k*y) \geq \alpha$. Then $A(k*x^{-1}) \vee m \geq A(k*x) \wedge n$. So

$$\begin{aligned} A(k*x^{-1}y) \vee m \vee m &\geq [(A(k*x^{-1}) \wedge A(k*y)) \wedge n] \vee m \\ &\geq ((A(k*x^{-1}) \vee m) \wedge (A(k*y)) \wedge n] \vee m) \\ &\geq ((A(k*x) \wedge n) \wedge (A(k*y)) \wedge n) \vee m) \\ &\geq ((A(k*x) \wedge n) \wedge (A(k*y)) \wedge n) \\ &\geq (A(k*x) \wedge A(k*y) \wedge n \\ &\geq \alpha \wedge \alpha \wedge n \\ &\geq \alpha. \end{aligned}$$

It follows that $x^{-1}y$ is in $A(\alpha)$ acted by K. Thus $A(\alpha)$ is an L-subgroup acted by K for any $\alpha \in [m,n]$ and $A(\alpha) \neq \{\ \}$.

Case (2): (ii) implies (i).

Let $A(\alpha)$ is an L-subgroup acted by K for any $\alpha \in [m,n]$ and $A(\alpha) \neq \{\ \}$.

First it need to check that $A(k*(xy)) \vee m \geq (A(k*x) \wedge A(k*y)) \wedge n$ for all k in K ,and x, y in the group S.Suppose this does not hold. Then there exists x_0, y_0 in S such that

$A(k*(x_0y_0)) \vee m = \alpha < (A(k*x_0) \wedge A(k*y_0)) \wedge n$.

$A(k*x_0) \geq \alpha$, $A(k*y_0) \geq \alpha$, and $\alpha \in [m,n]$. Further $x_0, y_0 \in A(\alpha)$. Then $A(k*(x_0^{-1}y_0)) < \alpha$, and so $x_0^{-1}y_0$ does not belong to $A(\alpha)$, which is impossible to fact that $A(\alpha)$ is an L-subgroup acted by K for any $\alpha \in [m,n]$. Concluded that $A(k*(xy)) \vee m \geq (A(k*x) \wedge A(k*y)) \wedge n$ for all k in K ,and x, y in S. Similarly $A(k*x^{-1}) \vee m \geq A(k*x) \wedge n$ for all $k \in K$, and $x, y \in S$. Therefore A is (m,n)-L-fuzzy subgroup acted by K. □

Theorem 3.4. *Let $f : S_1 \to S_2$ be a homomorphism on lattice ordered groups, and let A be (m,n)-L-fuzzy subgroup on S_1 acted by a non-empty set K. Then $f(A)$ is (m,n)-L-fuzzy subgroup of S_2, where $f(A)(k*y) = \vee\{A(k*x) : f(x) = y\}$.*

Proof. If $f^{-1}(k*x) = \{\ \}$ or $f^{-1}(k*y) = \{\ \}$, then $f(A)(k*x^{-1}y) \vee m \geq 0 = f(A)(k*x) \wedge f(A)(k*y) \wedge n$.

Otherwise $f^{-1}(k*x) \neq \{\ \}$ and $f^{-1}(k*y) = \{\ \}$ for all k in K, and x, y in S_2.

In this case,

$$f(A)(k*x^{-1}y)\vee m = [\vee\{A(k*t) : f(t) = x^{-1}y\}]\vee m \text{ where } t \in S_1.$$
$$\geq \vee\{A(k*a^{-1}b)\vee m : f(a) = x; f(b) = y\} \text{ where } a, b \text{ in } S_1.$$
$$\geq \vee(A(k*a)\wedge A(k*b))\wedge n : f(a) = x; f(b) = y\} \text{ where } a, b \text{ in } S_1.$$
$$\geq \vee\{A(k*a) : f(a) = x\}\wedge\{\vee\{A(k*b) : f(b) = y\}\wedge n = f(A)(k*x)\wedge f(A)($$

Similarly $f(A)(k*x^{-1})\vee m \geq f(A)(k*x)\wedge n$. Hence $f(A)$ is (m,n)-L-fuzzy subgroup of S_2. □

Theorem 3.5. *Let $f : S_1 \to S_2$ be a homomorphism on lattice ordered groups, and B be (m,n)-L-fuzzy subgroup on S_2 acted by a non-empty set K. Then $f^{-1}(B)$ is (m,n)-L-fuzzy subgroup of S_1, where $f^{-1}(B)(k*x) = B(k*f(x))$ for all x in S_1.*

Proof. Let x, y be in S_1.

Then $f^{-1}(B)(k*x^{-1}y)\vee m = B(k*f(x^{-1}y))\vee m = B(k*f(x^{-1})f(y))\vee m$
$= B(k*f(x)^{-1}f(y))\vee m$
$\geq B(k*f(x))\wedge B(k*f(y))\wedge n$
$= f^{-1}B(k*x)\wedge f^{-1}B(k*y)\wedge n$.

Similarly $f^{-1}(B)(k*x^{-1})\vee m \geq f^{-1}(B)(k*x)\wedge n$. Thus $f^{-1}(B)$ is (m,n)-L-fuzzy subgroup of S_1. □

Definition 3.1. *Let S_1 and S_2 be two lattice ordered groups with identities e_1 & e_2 respectively, both acted by a non-empty set K. Define $(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2)$ in $S_1\times S_2$. Note that (a_1,b_1) is the inverse of the element (a_2,b_2) if and only if a_1 is the inverse of a_2 in S_1, and b_1 is the inverse of b_2 in S_2.*

Theorem 3.6. *Let A and B be (m,n)-L-fuzzy subgroups of lattice ordered groups S_1 and S_2 respectively, both acted by a non-empty K. Then their product $A\times B$ is (m,n)-L-fuzzy subgroups of lattice ordered groups $S_1\times S_2$ acted by K, where $(A\times B)(k*(x,y)) = A(k*x)\wedge B(k*y)$.*

Proof. Let (a^{-1}, b^{-1}) be the inverse of (a,b) in $S_1\times S_2$. Then a^{-1} is the inverse of a in S_1, and b^{-1} is the inverse of b in S_2. Thus $A(k*a^{-1})\vee m \geq A(k*a)\wedge n$, and $A(k*b^{-1})\vee m \geq A(k*b)\wedge n$.

For all (a, b), and (x, y) in $S_1 \times S_2$, it gets that

$$\begin{aligned}
[(A \times B)(k * (a,b)^{-1}(x,y))] \vee m &= [(A \times B)(k * (a^{-1}, b^{-1})(x,y))] \vee m \\
&= [A(k * (a^{-1}x) \wedge B(k * (b^{-1}y)] \vee m \\
&= [A(k * (a^{-1}x) \vee m] \wedge [B(k * (b^{-1}y) \vee m] \\
&= [A(k * a) \wedge A(k * x) \wedge n] \wedge [B(k * b) \wedge B(k * y) \wedge n] \\
&= [A(k * a) \wedge B(k * b) \wedge [A(k * x) \wedge B(k * y)] \wedge n \\
&= (A \times B)(k * (a,b)] \wedge (A \times B)(k * (x,y)] \wedge n.
\end{aligned}$$

Similarly $(A \times B)(k * (a,b)^{-1}) \vee m \geq (A \times B)(k * (a,b)) \wedge n$.

Thus $A \times B$ is (m, n)-L-fuzzy subgroups of lattice ordered groups $S_1 \times S_2$ acted by K. □

Theorem 3.7. *Let A, B is (m, n)-L-fuzzy subgroups of lattice ordered groups S_1, and $S_2$with e_1 and e_2 as identity respectively, both acted by a non-empty K. If $A \times B$ is (m, n)-L-fuzzy subgroup of lattice ordered group $S_1 \times S_2$ acted by K, then one of the following statements must hold:*

(1) $A(k * e_1) \vee m \geq B(k * a) \wedge n$ for all a in S_2

(2) $B(k * e_2) \vee m \geq A(k * a) \vee n$ for all a in S_1.

Proof. Let $A \times B$ be (m, n)-L-fuzzy subgroup of lattice ordered group $S_1 \times S_2$ acted by a non-empty K. Suppose neither (1) nor (2) holds. Then there exist y in S_2 and x in S_1 with $A(k * e_1) \vee m < B(k * y) \wedge n,$, and $B(k * e_2) \vee m < A(k * x) \wedge n$. Then it gets that

$$\begin{aligned}
(A \times B)(k * (x,y)) \wedge n &= (A(k * x) \wedge B(k * y)) \wedge n \\
&= (A(k * x) \wedge n) \wedge (B(k * y) \wedge n) \\
&> (B(k * e_2) \vee m) \wedge (A(k * e_1) \vee m) \\
&= (A(k * e_1) \wedge (B(k * e_2)) \vee m \\
&> (A \times B)(k * (e_1, e_2)) \vee m.
\end{aligned}$$

Since $(A \times B)$ is (m, n)-L-fuzzy subgroups of lattice ordered groups $S_1 \times S_2$ acted by K, then $(A \times B)(k * (e_1, e_2)) \vee m \geq (A \times B)(k * (x, y)) \wedge n$ for all x in S_1, and y in S_2, which is a contradiction with the fact that (e_1, e_2) is the identity of $S_1 \times S_2$, and by (??). □

Theorem 3.8. *Let A, B is (m, n)-L-fuzzy subsets of lattice ordered groups S_1, and S_2 with e_1 and e_2 as identity respectively, both acted by a non-empty K. If $A \times B$ is (m, n)-L-fuzzy subgroups of lattice ordered groups $S_1 \times S_2$ acted by K satisfying $B(k * e_2) \vee m \geq A(k * a) \wedge n$ for all a in S_1, then A is (m, n)-L-fuzzy subgroup of lattice ordered group S_1 acted by K.*

Proof. From the assumption $B(k * e_2) \vee m \geq A(k * a) \wedge n$ for all a in S_1. If $B(k * e_2) \vee m \leq A(k * a)$ for some a in S_1, and $B(k * e_2) \vee m \leq n$, then $B(k * e_2) \vee m \leq A(k * a) \wedge n$, a contradiction. Then It obtains that either $B(k * e_2) \vee m \leq n$ (or) $B(k * e_2) \vee m \geq A(k * a)$ for all a in S_1.

Let x, y be in S_1. Then (x, e_2), and (y, e_2) are in $S_1 \times S_2$.

Case (1). Let $B(k*e_2) \vee m \geq n$ implies that . $B(k*e_2) \vee m \geq n \geq A(k*a) \wedge n$ for all a in S_1.

Then

$$
\begin{aligned}
&A(k * xy) \vee m \geq A(k * xy) \vee B(k * e_2e_2) \vee m \vee m \vee m \\
&= (A \times B)(k * (xy, e_2e_2)) \vee m \vee m \\
&\geq (A \times B)((k * (x, e_2))(k * (y, e_2))) \vee m \vee m \vee m \\
&= [(A \times B)((k * (x, e_2))(k * (y, e_2))) \vee m] \vee m \\
&\geq [(A \times B)(k * (x, e_2)) \wedge (A \times B)(k * (y, e_2)) \wedge n] \vee m \vee m \\
&\geq [(A \times B)(k * (x, e_2)) \vee m] \wedge [(A \times B)(k * (y, e_2)) \vee m] \wedge (n \vee m) \\
&\geq [A(k * x) \wedge B(k * e_2) \wedge n] \wedge [A(k * y) \wedge B(k * e_2) \wedge n] \vee m \\
&\geq [A(k * x) \vee m] \wedge [B(k * e_2) veem] \wedge [A(k * y) \vee m] \wedge B(k * e_2) \vee m] \\
&\geq [A(k * x) \vee m] \wedge [A(k * x) \wedge n] \wedge [A(k * y) \vee m] \wedge [A(k * y) \wedge n] \\
&\geq A(k * x) \wedge [A(k * x) \wedge n] \wedge A(k * y) \wedge [A(k * y) \wedge n] \\
&\geq A(k * x) \wedge A(k * y) \wedge n.
\end{aligned}
$$

Similarly $A(k * x^{-1}) \geq A(k * x)$ for all x in S_1.

Hence A is (m, n)-L-fuzzy subgroup of lattice ordered group S_1 acted by K.

Case (2). Let $B(k * e_2) \vee m \geq A(k * a) \geq A(k * a) \wedge n$ for all a in S_1. By the same argument as previous steps, it gets that $A(k * xy) \vee m \geq A(k * x) \wedge A(k * y) \wedge n$, and $A(k * x^{-1}) \geq A(k * x)$ for all x in S_1. Hence A is (m, n)-L-fuzzy subgroups of lattice ordered groups S_1 acted by K. □

Theorem 3.9. *Let A, B is (m, n)-L-fuzzy subsets of lattice ordered groups S_1, and S_2 with e_1 and e_2 as identity respectively, both acted by a non-empty*

K. If $A \times B$ is (m,n)-L-fuzzy subgroups of lattice ordered groups $S_1 \times S_2$ acted by K satisfying $A(k * e_1) \vee m \geq B(k * a) \wedge n$ for all a in S_2 for all a in S_1, then B is (m,n)-L-fuzzy subgroups of lattice ordered groups S_1 acted by K.

4 (m,n)-L-fuzzy index with fuzzy group acted by a set

Definition 4.1. *Let A be (m,n)-L-fuzzy subgroup of lattice ordered group S acted by K, and x be an element of S whose identity element is e. The (m,n)-fuzzy order of x under A is the smallest positive integer t with $A(k*x^t) \vee m = A(k * e) \wedge n$ for all k in K. Here $O(A(k * x) = t$.*

Definition 4.2. *Let A be (m,n)-L-fuzzy subgroup of lattice ordered group S acted by K, Then A is (m,n)-L-fuzzy normal of S acted by K if $A(k*xy) = A(k * yx)$ for all $x, y \in S$, and $k \in K$.*

Definition 4.3. *Let A be (m,n)-L-fuzzy subgroup of lattice ordered group S acted by K, Then a (m,n)-L-fuzzy left coset of A under s in S is defined an (m,n)-L-fuzzy subset satisfying $[sA](k * x) \vee m = A(k * (s^{-1}x)) \wedge n$ for all x, s in S, and k in K.*

Similarly a (m,n)-L-fuzzy right coset of A under s in S is defined an (m,n)-L-fuzzy subset satisfying $[sA](k * x) \vee m = A(k * (xs^{-1})) \wedge n$ for all x, s in S, and k in K.

Theorem 4.1. *Let A be (m,n)-L-fuzzy subgroup of lattice ordered group S acted by a non-empty K, and $x \in S$. Then $A(k * x^t) \vee m \geq A(k * x) \wedge n$ for all positive integer t.*

Proof. It is derived by induction on t.Since A is a (m,n)-L-fuzzy subgroup of lattice ordered group S acted by K, it gets that $A(k * x_2) \vee m \geq A(k * x) \wedge A(k * x) \wedge n = A(k * x) \wedge n$. $A(k * x^3) \vee m \geq A(k * x^2) \wedge A(k * x) \wedge n \geq A(k * x) \wedge A(k * x) \wedge n = A(k * x) \wedge n$. This is true for $t = 2$, 3.Assume that $A(k*x^t) \vee m \geq A(k*x) \wedge n$ where t is a positive integer. Since A is a (m,n)-L-fuzzy subgroup of S acted by K, $A(k*x^{t+1}) \vee m \geq A(k*x^t) \wedge A(k*x) \wedge n \geq A(k * x) \wedge n \wedge A(k * x) \wedge n = A(k * x) \wedge n$.

Hence $A(k * x^t) \vee m \geq A(k * x) \wedge n$ for all positive integer t. □

Theorem 4.2. *Let A be (m, n)-L-fuzzy subgroup of lattice ordered group S acted by a non-empty K, and $x \in S$ be of finite (m, n)-L-fuzzy order t. If an integer $r > 0$ is relatively prime to t, then $A(k * x^r) \vee m = A(k * x) \wedge n$, and $A(k * x) \vee m = A(k * x) \wedge n$.*

Proof. Given that r, t are relatively primes. Then there exist integers a, b with $1 = ar + bt$.
Then

$$\begin{aligned}
A(k * x) \vee m &= A(k * x^{ar+bt}) \vee m = A(k * x^{ar+b}) \vee m \vee m \\
&= [A(k * (x^{ar} x^{bt})) \vee m] \vee m \geq [A(k * x^{ar}) \wedge A(k * x^{bt}) \wedge n] \vee m \\
&= [A(k * x^{ar}) \vee m] \wedge [A(k * x^{bt}) \wedge m] \wedge [n \vee m] \\
&\geq [A(k * x^r) \wedge n] \wedge [A(k * e) \wedge n]. \wedge [n \vee m] \\
&= [A(k * x^r) \wedge n]. \wedge n \\
&= [A(k * x^r) \wedge n].
\end{aligned}$$

$$\begin{aligned}
A(kx) \vee m \quad &= [A(k * x) \vee m] \vee m \\
&= [A(k * x^r) \vee n]. \vee m \\
&= [A(k * x^r) \vee m] \wedge (n. \vee m) \\
&\geq [A(k * x) \wedge n] \wedge n \\
&\geq [A(k * x) \wedge n].
\end{aligned}$$

Now $n \leq m$ implies that $A(k * x) \wedge n \leq A(k * x) \wedge m \leq A(k * x) \vee m$.
Hence $A(k * x) \vee m = A(k * x) \wedge n$.
Also $A(k * x^r) \vee m \geq A(k * x) \wedge n$ for all positive integer r by (??).
$A(k * x) \wedge m \geq [A(k * x^r) \wedge n]$ by above argument.
Therefore $A(k * x) \wedge n \leq n \leq m \leq A(k * x^r) \vee m$.
Thus $A(k * x^r) \vee m = A(k * x) \wedge n$ for any positive integer r. □

Theorem 4.3. *Let A be (m, n)-L-fuzzy subgroup of lattice ordered group S acted by a non-empty K, and $x \in S$. If $A(k * x^r) \vee m = A(k * e) \wedge n$ for some positive integer r, $O(A(k * x)$ divides r.*

Proof. Let $O(A(k * x)) = t$. Then $A(k * x^t) \vee m = A(k * e) \wedge n$. By Euclidean algorithm, there exist integers a, b with $r = qt + b$; $0 \leq b < t$.

Thus

$$
\begin{aligned}
A(k*x^b)\vee m = A(k*x^{r-qt})\vee(m\vee m) \\
&= [A(k*(x^r x^{-qt}))\vee m]\vee m \\
&\geq [[A(k*x^r)\wedge A(k*x^{-qt})]\wedge n]\vee m \\
&= [A(k*x^r)\vee m]\wedge[A(k*x^{-qt})\vee m]]\wedge(n\vee m) \\
&= [A(k*x^r)\vee m]\wedge[A(k*e^{-q})\vee m]\wedge(n\vee m) \\
&= [A(k*x^r)\vee m]\wedge[A(k*e)\vee m]\wedge(n\vee m) \\
&\geq [A(k*e)\wedge n]\wedge[A(k*e)\wedge n]\wedge n \\
&= A(k*e)\wedge n\wedge n \\
&= A(k*e)\wedge n
\end{aligned}
$$

In fact, $A(k*e)\wedge n \geq A(k*x^b)\vee m$. Thus $A(k*x^b)\vee m = A(k*e)\wedge n$, and so $b=0$.

Therefore $r=qt$ implies that $t=O(A(k*x)$ divides r. □

Theorem 4.4. *Let A be (m,n)-L-fuzzy subgroup of lattice ordered group S acted by a non-empty K. Let $x\in S$ with $t=O(A(k*x))$. If a positive integer q is relatively prime to t, then $A(k*x)\vee m \geq A(k*x^q)\wedge n \geq A(k*x)\wedge n$.*

Proof. Since q and t are relatively primes, there exist integers a,b with $1=aq+bt$. So it gets that

$$
\begin{aligned}
A(k*x)\vee m &= [A(k*(x^{aq+bt}))\vee m]\vee m \\
&= [A(k*(x^{aq}x^{bt}))\vee m]\vee m \\
&\geq [A(k*x^{aq})\wedge A(k*x^{bt})\wedge n]\vee m \\
&= [A(k*x^{aq})\vee m]\wedge[A(k*e^b)\vee m]\wedge(nveem) \\
&\geq [A(k*x^q)\wedge n]\wedge[A(k*e)\vee m]\wedge(n\vee m) \\
&\geq [A(k*x^q)\wedge n]\wedge[A(k*x^q)\wedge n]\wedge(n\vee m) \\
&\geq [A(k*x^q)\wedge n]\wedge(n\vee m) \\
&\geq [A(k*x^q)\vee m \\
&\geq [A(k*x)\wedge n \text{ by (??)}.
\end{aligned}
$$

□

Theorem 4.5. *Let A be (m,n)-L-fuzzy normal subgroup of lattice ordered group S acted by a non-empty K. Then $O(A(k*x))=O(A(k*y^{-1}xy))$ for all x,y in S.*

Proof. Let x, y be in the lattice ordered group S.

Let $t = O(A(k * x))$. Then $A(k * x^t) \vee m = A(k * e) \wedge n$.

Then

$$\begin{aligned} A(k * x^r) \vee m &= A(k * x^r e) \wedge m \\ &= A(k * (x^r y^{-1} y)) \vee m \\ &= A(k * y^{-1} x^r y) \wedge n \text{ since } A \text{ is normal in } S. \\ &= A(k * (y^{-1} xy)^r) \wedge n \end{aligned}$$

So $A(k * x^r) \vee m = A(k * (y^{-1}xy)^r) \wedge n$ for any positive integer r.

In particular,$A(k * (y^{-1}xy)^t) \vee m = A(k * e) \wedge n$. Thus $O(A(k * y^{-1}xy))$ divides $t = O(A(k * x))$.

If $O(A(k * y^{-1}xy)) = p$, then $A(k * (y^{-1}xy)^p) = A(k * e) \wedge n$ implies $A(k * x^p) \vee m = A(k * e) \wedge n$.

Thus $O(A(k * x))$ divides $p = O(A(k * y^{-1}xy))$.

Hence $O(A(k * x)) = O(A(k * y^{-1}xy))$ for all x, y in S. □

Theorem 4.6. *Let A be (m, n)-L-fuzzy normal subgroup of lattice ordered group S acted by a non-empty K. For x, y in S with $A(k*x) \vee m \neq A(k*y) \wedge n$, then $A(k * xy) \vee m = A(k * x) \wedge A(k * y) \wedge n$.*

Proof. Without loss of generality, assume that $A(k * x) \vee m \leq A(k * y) \wedge n$. Then

$$\begin{aligned} A(k * x) \vee m &= A(k * y^{-1} yx) \vee m \vee m \\ &= [A(k * yxy^{-1}) \wedge n] \vee m \\ &= [[A(k * (yxy^{-1})) \vee m] \wedge [n \vee m]] \vee m \\ &\geq [A(k * y) \wedge A(k * xy^{-1}) \wedge n \wedge (n \vee m)] \vee m \\ &\geq A(k * y) \vee m] \wedge [A(k * xy^{-1}) \wedge m] \wedge (n \vee m) \wedge (n \vee m) \\ &\geq [A(k * y) \wedge n] \wedge [A(k * x) \wedge A(k * y) \wedge n] \\ &\geq A(k * x) \wedge A(k * y) \wedge n. \end{aligned}$$

So $A(k * x) \wedge m \geq A(k * x) \geq A(k * x) \wedge n$.

$A(k * x) \wedge A(k * y) \wedge n = [A(k * x) \vee m] \wedge A(k * y)$ □

Theorem 4.7. *Let A be (m, n)-L-fuzzy normal subgroup of lattice ordered group S acted by a non-empty K. Then the statements mentioned below are equivalent:*

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(1) A is normal in S.

(2) $A(k * y^{-1}xy) \vee m = A(k * x) \wedge n$ *for all* x, y *in* S.

(3) $A(k * xy) \vee m = A(k * yx) \wedge n$ *for all* x, y *in* S.

Proof. Case (i) - (1) → (2): □

References

[1] Cho, Y.U. and Jun, Y.B., On intuitionistic fuzzy R-subgroup of near - rings, Journal of Appl. Math and computing, Volume 18, (2005), 665–677.

[2] Gu, W.X., Li, S.Y., and Chen, D.G., Fuzzy groups with operators, fuzzy sets and system,66 (1994) ,363–371.

[3] Goguen, J.A., L-Fuzzy Sets, J. Math Anal.Appl. 18, 145-174 (1967).

[4] Kim, K.H., and Jun, Y.B., On fuzzy R-subgroups of near - rings, J. Fuzzy math, Volume 8, (2000), 549–558.

[5] Kim, K.H., and Jun, Y.B., Normal fuzzy R- subgroups of near - rings, J. fuzzy sets and system, Volume 121, (2001), 341–345.

[6] Kim, K.H., and Jun, Y.B., A note on fuzzy R-subgroups of near-rings, Soochow Journal of Mathematics, Volume 28, (2002), 339–346.

[7] Osman Kazanci, Sultar Yamark and Serife Yimaz, On intuitionistic Q-fuzzy R- subgroups of near- rings, International mathematical forum, Volume 59, (2007), 2899–2910 .

[8] Satya Saibaba. G.S.V., Fuzzy lattice ordered groups, South east Asian Bulletin of Mathematics Volume 32,(2008), 749–766.

[9] Solairaju, A., and Nagarajan, R., Q-fuzzy left R-subgroup of near-rings with respect to T- norm, Antarctica Journal of Mathematics, Volume 5, No. 1-2, (2008), 59–63.

[10] Solairaju and Nagarajan,R., Lattice Valued Q-fuzzy left R-submodules of neat-rings with respect to T, S-norms, Advances in fuzzy mathematics 4(2), 137–145 (2009).

[11] Solairaju, A., and Nagarajan, R., A New structure and constructions of Q - fuzzy groups, advances in fuzzy mathematics. Volume 4, (2009), 23–29.

[12] Solairaju, A., and Nagarajan, R., Some structure properties of upper Q- fuzzy index order with upper Q- fuzzy subgroups, Int. J. of open problems in mathematics and applications, Volume 1, (2011), 21–28.