Intuitionistic Fuzzy implicative filters of Lattice pseudo-Wajsberg Algebras

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Abstract—In this paper, we introduce the notion of an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra and investigate some properties with illustrations. Further, we obtain some equivalent conditions of an intuitionistic fuzzy implicative filter in lattice pseudo-Wajsberg algebra.

Keywords—Wajsberg algebra; Pseudo-Wajsberg algebra; Implicative filter; Fuzzy implicative filter; Intuitionistic fuzzy implicative filter.

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1. INTRODUCTION

Generalization of fuzzy sets leads to intuitionistic approach, that can further utilized to relate the membership and non-membership function introduced by Atanassov [1] in 1986. Every intuitionistic approach is explained in terms of upper and lower level sets. In 1965, Zadeh [10] extended the notion of binary membership to accommodate various “degrees of membership” on the real continuous interval [0,1]. Fuzzy sets and intuitionistic fuzzy sets are two strong frameworks for uncertainty handling. Basheer Ahamed and Ibrahim [2] introduced the definition of fuzzy implicative filters of lattice Wajsberg algebras and obtained some properties with illustrations. Pseudo-Wajsberg algebras are generalizations of Wajsberg algebras. Pseudo-Wajsberg algebras were introduced by Rodica Ceterchi [7] with the explicit purpose of providing a concept categorically equivalent to that of pseudo-MV algebras. Recently, the authors [5, 6] introduced the notions of implicative filter, fuzzy implicative filter of lattice pseudo-Wajsberg algebra and discussed some of their properties.

The aim of this paper is to introduce the definition of an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra and we obtain some related properties and equivalent conditions. We show that characterizations of an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra.

II. PRELIMINARIES

In this section, we give some elementary results that are necessary for our main results.

Definition 2.1[7]. An algebra \((A, \rightarrow, \bowtie, ^\sim, ~, 1)\) with a binary operations "\(\rightarrow\)" and "\(\bowtie\)" and quasi complements "\(^\sim\)" and "\(\sim\)" is called a pseudo-Wajsberg algebra if it satisfies the following axioms for all \(x, y, z \in A\),

(i) \(a\) \(1 \rightarrow x = x\)
\(b\) \(1 \bowtie x = x\)

(ii) \((x \bowtie y) \rightarrow y = (y \bowtie x) \rightarrow x = (y \rightarrow x) \bowtie x = (x \rightarrow y) \bowtie y\)

(iii) \(a\) \((x \rightarrow y) \rightarrow ((y \rightarrow z) \bowtie (x \rightarrow z)) = 1\)
\(b\) \((x \bowtie y) \bowtie ((y \bowtie z) \rightarrow (x \bowtie z)) = 1\)

(iv) \(1^\sim = 1^- = 0\)

(v) \(a\) \((x^\sim \bowtie y^\sim) \rightarrow (y \rightarrow x) = 1\)
\(b\) \((x^\sim \rightarrow y^\sim) \bowtie (y ^\sim \bowtie x) = 1\)
(vi) \((x \rightarrow y^-)^- = (y \bowtie x^-)^-\).

**Definition 2.2**[7]. An algebra \((A, \rightarrow, \bowtie, \neg, \Rightarrow, 1)\) is called a lattice pseudo-Wajsberg algebra if it satisfies the following axioms for all \(x, y \in A\),

(i) A partial ordering “\(\leq\)” on a lattice pseudo-Wajsberg algebra \(A\), such that \(x \leq y\) if and only if \(x \rightarrow y = 1\) if and only if \(x \bowtie y = 1\).

(ii) \(x \vee y = (x \rightarrow y) \bowtie y = (y \rightarrow x) \bowtie x = (x \bowtie y) \rightarrow y = (y \bowtie x) \rightarrow x\)

(iii) \(x \wedge y = (x \bowtie (x \rightarrow y))^- = ((x \rightarrow y) \rightarrow x^-)^- = (y \bowtie (y \rightarrow x))^- = ((y \rightarrow x) \rightarrow y^-)^- = (y \bowtie (x \rightarrow y))^- = (x \bowtie (x \rightarrow y))^- = ((x \bowtie y) \bowtie x^-)^-\)

**Proposition 2.3**[3]. A Wajsberg algebra \((A, \rightarrow, \neg, 1)\) satisfies the following axioms for all \(x, y \in A\),

(i) \(x \rightarrow x = 1\)

(ii) If \(x \rightarrow y = y \rightarrow x = 1\), then \(x = y\)

(iii) \(x \rightarrow 1 = 1\)

(iv) \(x \rightarrow (y \rightarrow x) = 1\)

(v) If \(x \rightarrow y = y \rightarrow z = 1\), then \(x \rightarrow z = 1\)

(vi) \((x \rightarrow y) \rightarrow ((x \rightarrow z) \rightarrow (y \rightarrow z)) = 1\)

(vii) \(x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)\).

**Proposition 2.4**[7]. A lattice pseudo-Wajsberg algebra \((A, \rightarrow, \bowtie, \neg, \Rightarrow, 1)\) satisfies the following axioms for all \(x, y \in A\),

(i) \(x \rightarrow x = 1, x \bowtie x = 1\)

(ii) \(x \rightarrow (y \bowtie x) = 1, x \bowtie (y \rightarrow x) = 1\)

(iii) \(x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y; z \bowtie x \leq z \bowtie y\)

(iv) \(x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z; y \bowtie z \leq x \bowtie z\)

(v) \(x \leq y \Rightarrow x; x \leq y \bowtie x\)

(vi) \(x \rightarrow y \leq (y \rightarrow z) \bowtie (x \rightarrow z)\); \(x \bowtie y \leq (y \bowtie z) \rightarrow (x \bowtie z)\)

(vii) \(x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)\); \(x \bowtie y \leq (z \bowtie x) \bowtie (z \bowtie y)\)

(viii) \(x \rightarrow (y \bowtie z) = y \bowtie (x \rightarrow z)\).

**Definition 2.5**[5]. Let \(A\) be lattice pseudo-Wajsberg algebra. A non empty subset \(F\) of \(A\) is called an implicative filter of \(A\) if it satisfies the following axioms for all \(x, y \in A\),

(i) \(1 \in F\)

(ii) \(x \in F\) and \(x \rightarrow y \in F\) imply \(y \in F\)

(iii) \(x \in F\) and \(x \bowtie y \in F\) imply \(y \in F\).

**Proposition 2.6**[5]. Let \(F\) be any implicative filter of lattice pseudo-Wajsberg algebra of \(A\) satisfying the axiom \(x \bowtie (y \bowtie z) = y \bowtie (x \bowtie z)\) for all \(x, y \in A\).

**Definition 2.7**[1]. Let \(A\) be a set. A function \(\mu: A \rightarrow [0, 1]\) is called a fuzzy subset on \(A\) for each \(x \in A\), the value of \(\mu(x)\) describes a degree of membership of \(x\) in \(\mu\).

**Definition 2.8**[10]. Let \(\mu\) be a fuzzy subset of \(A\) then the complement of \(\mu\) is denoted by \(\overline{\mu(x)}\) and defined as \(\overline{\mu(x)} = 1 - \mu(x)\) for all \(x \in A\).

**Definition 2.9**[6]. Let \(A\) be lattice pseudo-Wajsberg algebra. A non empty fuzzy subset \(\mu\) of \(A\) is called an implicative filter of \(A\) if it satisfies the following axioms for all \(x, y \in A\),

(i) \(\mu(1) \geq \mu(x)\)

(ii) \(\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\}\)

(iii) \(\mu(y) \geq \min\{\mu(x \bowtie y), \mu(x)\}\).
Definition 2.10[1]. An intuitionistic fuzzy set $T$ of a non-empty set $A$ is an object having the form

$$T = \{ (x, \mu_t(x), \gamma_t(x)) \mid x \in A \} = (\mu_t, \gamma_t)$$

where the functions $\mu_t(x) : A \to [0,1]$ and $\gamma_t(x) : A \to [0,1]$ denote the degree of membership and the degree of non-membership respectively, and $0 \leq \mu_t(x) + \gamma_t(x) \leq 1$ for any $x \in A$.

III. MAIN RESULTS

In this section, we introduce an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra and investigate some properties with illustrations.

Definition 3.1. Let $A$ be lattice pseudo-Wajsberg algebra. An intuitionistic fuzzy set $T = (\mu_t, \gamma_t)$ of $A$ is called an intuitionistic fuzzy implicative filter of $A$ if it satisfies the following axioms for all $x, y \in A$,

(i) $\mu_t(1) \geq \mu_t(x) ; \gamma_t(1) \leq \gamma_t(x)$
(ii) $\mu_t(1) \geq \min \{ \mu_t(x \rightarrow y), \mu_t(x) \} ; \gamma_t(1) \leq \max \{ \gamma_t(x \rightarrow y), \gamma_t(x) \}$
(iii) $\mu_t(x) \leq \min \{ \mu_t(x \leftarrow y), \mu_t(x) \} ; \gamma_t(x) \leq \max \{ \gamma_t(x \leftarrow y), \gamma_t(x) \}$

Example 3.2 Consider a set $A = \{ 0, a, b, c, 1 \}$. Define a partial ordering “$\leq$” on $A$ such that $0 < a < b , c < 1$ and the binary operations “$\rightarrow$”, “$\leftarrow$” and quasi complements “$\Rightarrow$”, “$\leadsto$” given by the following tables 3.1, 3.2, 3.3 and 3.4.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$x^-$</th>
<th>$\rightarrow$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$1$</th>
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<tr>
<td>0</td>
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<tr>
<th>$x$</th>
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<th>$\Rightarrow$</th>
<th>$a$</th>
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</table>

Consider an intuitionistic fuzzy subset $T = (\mu_t, \gamma_t)$ on $A$ as, $\mu_t(x) = \begin{cases} 0.2 & \text{if } x = 1 \\ 0.5 & \text{otherwise} \end{cases}$ for all $x \in A$

$\gamma_t(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.3 & \text{otherwise} \end{cases}$ for all $x \in A$

Then $T$ is an intuitionistic fuzzy implicative filter of $A$.

In the same Example 3.2, let us consider an intuitionistic fuzzy subset $T = (\mu_t, \gamma_t)$ on $A$ as,

$\mu_t(x) = \begin{cases} 0.41 & \text{if } x \in \{ 0, b, c \} \\ 0.23 & \text{otherwise} \end{cases}$ for all $x \in A$

$\gamma_t(x) = \begin{cases} 0.51 & \text{if } x \in \{ 0, b, c \} \\ 0.38 & \text{otherwise} \end{cases}$ for all $x \in A$

Then $T$ is not an intuitionistic fuzzy implicative filter of $A$.

Since $\mu_t(a) \leq \min \{ \mu_t(c \rightarrow a), \mu_t(c) \}$ and $\gamma_t(a) \leq \max \{ \gamma_t(c \rightarrow a), \gamma_t(c) \}$. 

Proposition 3.3. Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$, then $x \leq y$ implies $\mu_t(x) \leq \mu_t(y)$ and $\gamma_t(x) \geq \gamma_t(y)$ for all $x, y \in A$. 


Proof. Let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of \( A \). If \( x \leq y \) then \( x \rightarrow y = x \sim y = 1 \) for all \( x, y \in A \) and, so \( \mu_t(y) \geq \min(\mu_t(x \rightarrow y), \mu_t(x)) = \min(\mu_t(x \sim y), \mu_t(x)) \geq \min \{ \mu_t(1), \mu_t(x) \} = \mu_t(x) \) for all \( x, y \in A \).

\[
\gamma_t(y) \leq \max \{ \gamma_t(x \rightarrow y), \gamma_t(x) \} = \max \{ \gamma_t(x \sim y), \gamma_t(x) \} \leq \max(\gamma_t(1), \gamma_t(x)) = \gamma_t(x) \) for all \( x, y \in A \).
\]

Thus \( \mu_t(x) \leq \mu_t(y) \) and \( \gamma_t(y) \leq \gamma_t(x) \) for all \( x, y \in A \).

\[ (3.1) \]

\[ (3.2) \]

**Proposition 3.4.** Let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra \( A \) if and only if \( x \leq y \rightarrow z \) and \( x \leq y \sim z \) implies \( \mu_t(z) \geq \min \{ \mu_t(x), \mu_t(y) \} \)

\[ x \geq y \rightarrow z \text{ and } x \geq y \sim z \text{ implies } \gamma_t(z) \leq \max(\gamma_t(x), \gamma_t(y)) \text{ for all } x, y, z \in A. \]

Proof. Let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of \( A \) satisfying the condition (3.1).

If \( x \leq y \rightarrow z \) and \( x \leq y \sim z \) then \( \mu_t(1) = \min(\mu_t(x), \mu_t(y)) \)

\[
\mu_t(x) \leq \min(\mu_t(x \sim y), \mu_t(x)) \leq \min(\mu_t(1), \mu_t(x)) = \mu_t(x) \]

Hence, \( T = (\mu_t, \gamma_t) \) is an intuitionistic fuzzy implicative filter of \( A \).

Conversely, let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of \( A \),

If \( x \leq y \rightarrow z \) and \( x \leq y \sim z \), then \( \mu_t(x) \leq \mu_t(y \rightarrow z) \) and \( \mu_t(x) \leq \mu_t(y \sim z) \)

[From Proposition 3.3.]

From (ii) and (iii) of definition 3.1, we have

\[
\mu_t(z) \geq \min(\mu_t(x \rightarrow y), \mu_t(y)) \geq \min(\mu_t(x \sim y), \mu_t(y)) \geq \min(\mu_t(1), \mu_t(y)) = \mu_t(y) \]

Hence \( x \leq y \rightarrow z \) and \( x \leq y \sim z \) implies \( \mu_t(z) \geq \min \{ \mu_t(x), \mu_t(y) \} \) for all \( x, y, z \in A \).

Similarly, \( \gamma_t(x) \leq \gamma_t(y \rightarrow z) \) and \( \gamma_t(x) \leq \gamma_t(y \sim z) \)

[From Proposition 3.3.]

From (ii) and (iii) of definition 3.1, we have

\[
\gamma_t(z) \leq \max(\gamma_t(x \rightarrow y), \gamma_t(y)) \leq \max(\gamma_t(x \sim y), \gamma_t(y)) \leq \max(\gamma_t(1), \gamma_t(y)) = \gamma_t(y) \]

Hence \( x \geq y \rightarrow z \) and \( x \geq y \sim z \) implies \( \gamma_t(z) \leq \max(\gamma_t(x), \gamma_t(y)) \) for all \( x, y, z \in A \).

**Proposition 3.5.** Let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra \( A \) if and only if the fuzzy sets \( \mu_t \) and \( \gamma_t \) are implicative filters of lattice pseudo-Wajsberg algebra \( A \), where \( \gamma_t(x) = 1 - \gamma_t(x) \).

Proof. Let \( T = (\mu_t, \gamma_t) \) be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra \( A \). We know that the fuzzy set \( \mu_t \) is a fuzzy implicative filter of lattice pseudo-Wajsberg algebra \( A \).

\[
\gamma_t(1) = 1 - \gamma_t(1) \geq 1 - \gamma_t(x) = 1 - \gamma_t(x) \]

[From the definition 2.8]

\[
\gamma_t(y) = 1 - \gamma_t(y) \geq 1 - \gamma_t(x) \]

[From the definition 3.1]

Similarly, we prove \( \gamma_t(y) \)

[From the definition 2.8]

Therefore \( \gamma_t \) is a fuzzy implicative filter of lattice pseudo-Wajsberg algebra.

Conversely, let \( \mu_t, \gamma_t \) be a fuzzy implicative filter of lattice pseudo-Wajsberg algebra.

Let \( \mu_t(1) \geq \mu_t(x) \) and \( \gamma_t(1) \leq \gamma_t(x) \) for all \( x \in A \)

[From (i) of definition 3.1]

We have \( \gamma_t(1) \geq \gamma_t(x) \) and \( \mu_t(y) \geq \min(\mu_t(x \rightarrow y), \mu_t(x)) \) and \( \mu_t(y) \geq \min(\mu_t(x \sim y), \mu_t(x)) \) for all \( x, y \in A \)

[From (ii) and (iii) of definition 3.1]

Now \( \gamma_t(y) = \min(\gamma_t(x \rightarrow y), \gamma_t(x)) \)
1 − γ(x → y) ≥ \min\{1 − γ(x → y), 1 − γ(x)\}
\gamma(x) = 1 − \max\{\gamma(x → y), \gamma(x)\}
\gamma(y) ≤ \max\{\gamma(x → y), \gamma(x)\}
for all \(x, y \in A\).
Similarly, we prove that \(\gamma(y) \leq \max\{\gamma(x \sim y), \gamma(x)\}\) for all \(x, y \in A\).
Therefore \(T = (\mu_t, \gamma_t)\) is an intuitionistic fuzzy implicational filter of \(A\).

**Proposition 3.6.** Let \(T = (\mu_t, \gamma_t)\) be an intuitionistic fuzzy implicational filter of lattice pseudo-Wajsberg algebra \(A\) if and only if \(T_1 = \{(x, \mu_t(x), \mu_t(x))\// x \in A\}\) and \(T_2 = \{(x, \gamma_t(x), \gamma_t(x))\// x \in A\}\) are both intuitionistic fuzzy implicational filters of lattice pseudo-Wajsberg algebra \(A\). Where \(\mu_t(x) = 1 - \mu_t(x), \gamma_t(x) = 1 - \gamma_t(x)\).

**Proof.** Let \(T = (\mu_t, \gamma_t)\) be an intuitionistic fuzzy implicational filter of lattice pseudo-Wajsberg algebra \(A\), then for any \(x \in A\), \(0 ≤ \mu_t(x) + \mu_t(x) = \mu_t(x) + 1 - \mu_t(x) = 1\)
Similarly, \(0 ≤ \gamma_t(x) + \gamma_t(x) = \gamma_t(x) + 1 - \gamma_t(x) = 1\).
Therefore \(T_1\) and \(T_2\) are intuitionistic fuzzy implicational filters of \(A\).
Let \(x, y \in A\), \(\mu_t(1) ≥ \mu_t(x), \mu_t(1) = 1 - \mu_t(1) ≤ 1 - \mu_t(x) = \mu_t(x)\)
We have \(\mu_t(y) \geq \min\{\mu_t(x → y), \mu_t(x)\}\) and \(\mu_t(y) \geq \min\{\mu_t(x → y), \mu_t(x)\}\)
\(\mu_t(y) = 1 - \mu_t(y) \leq 1 - \min\{\mu_t(x → y), \mu_t(x)\}\)
Similarly, we prove that \(\mu_t(y) = \max\{\mu_t(x → y), \mu_t(x)\}\)
Therefore \(T_1 = \{(x, \mu_t(x), \mu_t(x))\// x \in A\}\) is an intuitionistic fuzzy implicational filter of lattice pseudo-Wajsberg algebra \(A\).
Similarly, we prove that \(T_2 = \{(x, \gamma_t(x), \gamma_t(x))\// x \in A\}\) is an intuitionistic fuzzy implicational filter of lattice pseudo-Wajsberg algebra \(A\).
Conversely, if \(T_1 = \{(x, \mu_t(x), \mu_t(x))\// x \in A\}\) and \(T_2 = \{(x, \gamma_t(x), \gamma_t(x))\// x \in A\}\) are both intuitionistic fuzzy implicational filters of lattice pseudo-Wajsberg algebra \(A\) then

\begin{align*}
(1) & (a) \mu_t(x) ≥ \min\{\mu_t(x → y), \mu_t(x → y)\} \\
& (b) \gamma_t(x) ≤ \max\{\gamma_t(x → y), \gamma_t(x → y)\} \\
(2) & (a) \mu_t(x) ≥ \min\{\mu_t(x → y), \mu_t(x → y)\} \\
& (b) \gamma_t(x) ≤ \max\{\gamma_t(x → y), \gamma_t(x → y)\} \\
(3) & (a) \mu_t(x → y) ≥ \mu_t(x → (x → y)) \\
& (b) \gamma_t(x → y) ≤ \gamma_t(x → (x → y)) \\
& (c) \mu_t(x → y) ≥ \mu_t(x) ≥ \gamma_t(x → (x → y)) \\
& (d) \gamma_t(x → y) ≤ \gamma_t(x) ≤ \gamma_t(x → (x → y)) \\
(4) & (a) \mu_t(x → y) ≥ \mu_t(x → (x → y)) \\
& (b) \gamma_t(x → y) ≤ \gamma_t(x → (x → y)) \\
& (c) \mu_t(x → y) ≥ \mu_t(x → (x → y)) \\
& (d) \gamma_t(x → y) ≤ \gamma_t(x → (x → y)) \\
(5) & (a) \mu_t(x → y) ≥ \mu_t(x → (x → y)) \\
& (b) \gamma_t(x → y) ≤ \gamma_t(x → (x → y)) \\
& (c) \mu_t(x → y) ≥ \mu_t(x → (x → y)) \\
& (d) \gamma_t(x → y) ≤ \gamma_t(x → (x → y))
\end{align*}

**Proof.** (i)(a) ⇒ (ii)(a)
From (i)(a), let \(T = (\mu_t, \gamma_t)\) be an intuitionistic fuzzy implicational filter of lattice pseudo-Wajsberg algebra \(A\) and \(z = y, x = x\).
We have \(\mu_t(x → y) ≥ \min\{\mu_t(x → (x → y)), \mu_t(x → x)\}\) [From (ii) of definition 3.1]
Therefore, $\mu_t(x \to (x \to y)) \geq \mu_t(x \to (x \to y))$.

\[(i)(b) \Rightarrow (ii)(b)\]

From (i)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and $z = y, y = x$.

We have $\gamma_t(x \to y) \leq \max \{\gamma_t(x \to (x \to y)), \gamma_t(x \to x)\}$

$= \max \{\gamma_t(x \to (x \to y)), \gamma_t(1)\}$

From (i) of definition 3.1

Therefore, $\gamma_t(x \to y) \leq \gamma_t(x \to (x \to y))$.

Similarly, we show that (i)(c) $\Rightarrow$ (ii)(c) and (i)(d) $\Rightarrow$ (ii)(d)

\[(ii)(a) \Rightarrow (iii)(a)\]

From (ii)(a), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

Let $x \to (y \to z) \leq x \to ((x \to (y \to z)) \to (x \to z))$

It follows as $\mu_t(x \to (x \to (x \to y))) = \mu_t(x \to ((x \to (y \to z)))$.

\[(ii)(b) \Rightarrow (iii)(b)\]

From (ii)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

\[(iii)(a) \Rightarrow (i)(a)\]

From (iii)(a), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

\[(iii)(b) \Rightarrow (i)(b)\]

From (iii)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

\[
\begin{align*}
\mu_t(x \to z) & \geq \min \{\mu_t(x \to y) \to (x \to z), \mu_t(x \to y)\} & \text{[From (ii) of definition 3.1]} \\
\mu_t(x \to z) & \geq \min \{\mu_t(x \to (y \to z)), \mu_t(x \to y)\} & \text{[From (ii) of definition 3.1]}
\end{align*}
\]

\[(iii)(c) \Rightarrow (i)(c)\]

Similarly, we show that (iii)(c) $\Rightarrow$ (i)(c) and (iii)(d) $\Rightarrow$ (i)(d).

**Proposition 3.8.** Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ if and only if satisfies the following axioms

\[(i) \ (a) \ \mu_t(x) \geq \min \{\mu_t(y), \mu_t(x \to y)\}\]

\[(b) \ \gamma_t(x) \leq \max \{\gamma_t(y), \gamma_t(x \to y)\}\]

\[(ii) (a) \ \mu_t(x) \geq \min \{\mu_t(y), \mu_t(x \to y)\}\]

\[(b) \ \gamma_t(x) \leq \max \{\gamma_t(y), \gamma_t(x \to y)\}\]

for all $x, y \in A$.

**Proof.** (i)(a) Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and let $y \to 1 \leq (x \to y) \to (x \to 1)$

\[(vii) \text{ of proposition } 2.4\]

Thus, $\mu_t(y \to 1) \leq \mu_t(x \to y \to (x \to 1))$.

Consider $\mu_t(x \to 1) \geq \min \{\mu_t((x \to y) \to (x \to 1), \mu_t(x \to y)\}$.

\[(vii) \text{ of proposition } 2.4\]

We have $\mu_t(x \to 1) = \mu_t(x)$ and $\mu_t(y \to 1) = \mu_t(y)$

Therefore, $\mu_t(x) \geq \min \{\mu_t(y), \mu_t(x \to y)\}$

\[(vii) \text{ of proposition } 2.4\]

(i)(b) Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and let $y \to 1 \leq (x \to y) \to (x \to 1)$

\[(vii) \text{ of proposition } 2.4\]
Thus, $\gamma_t(y \to 1) \geq \gamma_t((x \to y) \to (x \to 1))$ [From the Proposition 3.3]

Consider $\gamma_t(x \to 1) \leq \max\{\gamma_t((x \to y) \to (x \to 1)), \gamma_t(x \to y)\}$ [From (ii) of definition 3.1]

Thus, $\gamma_t(x \to 1) \leq \max\{\gamma_t(y \to 1), \gamma_t(x \to y)\}$ [From (vii) of proposition 2.4]

We have $\gamma_t(x \to 1) = \gamma_t(x)$ and $\gamma_t(y \to 1) = \gamma_t(y)$

Hence, $\gamma_t(x) \leq \max\{\gamma_t(y), \gamma_t(x \to y)\}$

Similarly to prove $\mu_t(x) \geq \min\{\mu_t(y), \mu_t(x \sim y)\}$ and $\gamma_t(x) \leq \max\{\gamma_t(y), \gamma_t(x \sim y)\}$ for all $x, y \in A$.

The converse part is straightforward. ■

**Proposition 3.9.** Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. Then the following are equivalent for all $x, y, z \in A$.

(i) (a) $\mu_t(x) \geq \min \{\mu_t(z \to ((x \to y) \to x)), \mu_t(z)\}$

(b) $\gamma_t(x) \leq \gamma_t((x \to y) \to x)\]

(c) $\mu_t(x) \geq \min\{\gamma_t((x \to y) \to x)), \gamma_t(z)\}$

(d) $\gamma_t(x) \leq \max\{\gamma_t(z \to ((x \to y) \to x)), \gamma_t(z)\}$

(ii) (a) $\mu_t(x) \geq \min\{\mu_t((x \to y) \to x)\)$

(b) $\gamma_t(x) \leq \gamma_t((x \to y) \to x)$

(c) $\mu_t(x) \geq \gamma_t((x \to y) \to x)$

(d) $\gamma_t(x) \leq \gamma_t((x \to y) \to x)$

(iii) (a) $\mu_t(x) = \mu_t((x \to y) \to x)$

(b) $\gamma_t(x) = \gamma_t((x \to y) \to x)$

(c) $\mu_t(x) = \gamma_t((x \to y) \to x)$

(d) $\gamma_t(x) = \gamma_t((x \to y) \to x)$.

**Proof.** (i)(a) $\Rightarrow$ (ii)(a)

From (i)(a), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and $z = 1$ in (i)(a)

We have $\mu_t(x) \geq \min \{\mu_t(1 \to ((x \to y) \to x)), \mu_t(1)\} = \mu_t((x \to y) \to x)$

(i)(b) $\Rightarrow$ (ii)(b)

From (i)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$ and $z = 1$ in (i)(b)

We have $\gamma_t(x) \leq \gamma_t((x \to y) \to x)$

Similarly, we show that (i)(c) $\Rightarrow$ (ii)(c) and (i)(d) $\Rightarrow$ (ii)(d)

(ii)(a) $\Rightarrow$ (iii)(a)

From (ii)(a), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

Let $x \leq (x \to y) \to x$. We have, $\mu_t(x) \leq \mu_t((x \to y) \to x)$

It follows from (ii)(a) that $\mu_t(x) = \mu_t((x \to y) \to x)$

(ii)(b) $\Rightarrow$ (iii)(b)

From (ii)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

Let $x \leq (x \to y) \to x$.

We have $\gamma_t((x \to y) \to x) \geq \gamma_t(x)$

It follows from (ii)(b) that $\gamma_t(x) = \gamma_t((x \to y) \to x)$

Similarly, we show that (ii)(c) $\Rightarrow$ (iii)(c) and (ii)(d) $\Rightarrow$ (iii)(d)

(iii)(a) $\Rightarrow$ (i)(a)

From (iii)(a), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

We have, $\mu_t((x \to y) \to x) \geq \min \{\mu_t(z \to ((x \to y) \to x)), \mu_t(z)\}$

[From (ii) of definition 3.1]

Combining (iii)(a), we have $\mu_t(x) \geq \min \{\mu_t(z \to ((x \to y) \to x)), \mu_t(z)\}$

Hence $\mu_t(x) \geq \min \{\mu_t(z \to ((x \to y) \to x)), \mu_t(z)\}$
(iii) (b) ⇒ (i) (b)
From (iii)(b), let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.
We have $\gamma_t((x \rightarrow y) \rightarrow x) \leq \max \left\{ \gamma_t \left( z \rightarrow ((x \rightarrow y) \rightarrow x) \right), \gamma_t(z) \right\}$ \hspace{1cm} \text{[From (ii) of definition 3.1]}
Combining (iii)(b), we have $\gamma_t(x) \leq \max \left\{ \gamma_t \left( z \rightarrow ((x \rightarrow y) \rightarrow x) \right), \gamma_t(z) \right\}$
Hence $\gamma_t(x) \leq \max \left\{ \gamma_t \left( z \rightarrow ((x \rightarrow y) \rightarrow x) \right), \gamma_t(z) \right\}$
Similarly, we show that (iii) (c) ⇒ (i)(c) and (iii)(d) ⇒ (i)(d).

Proposition 3.10. Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. Then $T = (\mu_t, \gamma_t)$ satisfies the following axioms for all $x, y, z \in A$,

(i) (a) $\mu_t ((x \rightarrow y) \sim x) \geq \mu_t ((x \rightarrow y) \rightarrow y)$
(b) $\gamma_t ((x \rightarrow y) \sim x) \leq \gamma_t ((x \rightarrow y) \rightarrow y)$
(ii)(a) $\mu_t ((y \sim x) \rightarrow x) \geq \mu_t ((y \sim y) \sim y)$
(b) $\gamma_t ((y \sim x) \sim x) \leq \gamma_t ((y \sim y) \sim y)$

Proof (i)(a). Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. From (vii) and (viii) of proposition (2.4),
We have $((y \rightarrow x) \sim x, ((x \rightarrow y) \sim x) \rightarrow y \leq x \rightarrow y,$
$(x \rightarrow y) \sim (x \rightarrow y) \sim x) \leq (((x \rightarrow x) \rightarrow x) \rightarrow ((y \rightarrow x) \sim x))$
Hence, $(x \rightarrow y) \sim (x \rightarrow y) \sim x)$
We have $\mu_t ((x \rightarrow y) \rightarrow y) \leq \mu_t (((y \rightarrow x) \sim x) \sim (x \rightarrow y) \sim x)) \rightarrow ((y \rightarrow x) \sim x)) \hspace{1cm} \text{[From proposition 3.3]}
From (ii)(a) of proposition (3.9), we have $\mu_t (((y \rightarrow x) \sim x) \sim (x \rightarrow y) \sim x)) \hspace{1cm} \text{[From proposition 3.3]}
Hence, $\mu_t (((y \rightarrow x) \sim x) \sim (x \rightarrow y) \sim y))$.

(i)(b). Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. From (vi) and (viii) of proposition (2.4),
we have $((x \rightarrow y) \sim x, ((x \rightarrow y) \sim x) \rightarrow y \leq x \rightarrow y,$
$(x \rightarrow y) \sim (x \rightarrow y) \sim x) \leq (((x \rightarrow x) \sim x) \sim y) \rightarrow ((y \rightarrow x) \sim x))$
Hence $(x \rightarrow y) \sim (x \rightarrow y) \sim x)$
We have $\gamma_t ((x \rightarrow y) \sim x) \sim y) \rightarrow ((y \rightarrow x) \sim x)) \hspace{1cm} \text{[From the Proposition 3.3]}
We have $\gamma_t (((y \rightarrow x) \sim x) \sim y) \rightarrow ((y \rightarrow x) \sim x)) \rightarrow (y \rightarrow x) \sim x)) \hspace{1cm} \text{[From (ii) (b) of proposition 3.9]}
Hence $\gamma_t (((x \rightarrow x) \sim x) \sim y) \rightarrow y)$
Similarly, we proved $\mu_t ((y \sim x) \rightarrow x) \geq \mu_t ((x \sim y) \sim y)$ and $\gamma_t ((y \sim x) \rightarrow x) \sim y) \rightarrow y)$.

Proposition 3.11. An intuitionistic fuzzy subset $T = (\mu_t, \gamma_t)$ is an implicative filter of lattice pseudo-Wajsberg algebra $A$ if and only if
(i)(a) $\mu_t(x) = \mu_t((x \rightarrow y) \rightarrow x)$
(b) $\gamma_t(x) = \gamma_t((x \rightarrow y) \rightarrow x)$
(ii)(a) $\mu_t(x) = \mu_t((x \sim x) \sim x)$
(b) $\gamma_t(x) = \gamma_t((x \sim x) \sim x)$ for all $x, y, z \in A$.

Proof. (i)(a) Let $T = (\mu_t, \gamma_t)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. Consider $x \leq (x \rightarrow y) \rightarrow x$ and $T$ is an intuitionistic fuzzy implicative filter, $\mu_t(x) \leq \mu_t((x \rightarrow y) \rightarrow x)$
Since $T = (\mu_\tau, \gamma_\tau)$ is an intuitionistic fuzzy implicative filter,

$\mu_\tau(x) \geq \min \left\{ \mu_\tau \left( 1 \rightarrow ((x \rightarrow y) \rightarrow z) \right), \mu_\tau(1) \right\} = \mu_\tau((x \rightarrow y) \rightarrow z)$.

Hence $\mu_\tau(x) = \mu_\tau((x \rightarrow y) \rightarrow x)$.

(i)(b) Let $T = (\mu_\tau, \gamma_\tau)$ be an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$. Consider $(x \rightarrow y) \rightarrow x \leq x$ and is $T$ an intuitionistic fuzzy implicative filter, $\gamma_\tau((x \rightarrow y) \rightarrow x) \geq \gamma_\tau(x)$

[From the Proposition 3.3]

Since $T = (\mu_\tau, \gamma_\tau)$ is an intuitionistic fuzzy implicative filter,

$\gamma_\tau(x) \leq \max \left\{ \gamma_\tau \left( 1 \rightarrow ((x \rightarrow y) \rightarrow x) \right), \gamma_\tau(1) \right\} = \gamma_\tau((x \rightarrow y) \rightarrow x)$

Hence $\gamma_\tau(x) = \gamma_\tau((x \rightarrow y) \rightarrow x)$.

Similarly, we show that (ii)(a), (ii)(b).

Conversely, (i)(a), (i)(b), (ii)(a) and (ii)(b) holds.

Clearly, $\mu_\tau(1) \geq \mu_\tau(x)$ for all $x \in A$.

We have, $\mu_\tau(y) = \mu_\tau((y \rightarrow x) \rightarrow y) \geq \min \left\{ \mu_\tau((x \rightarrow y) \rightarrow x), \mu_\tau(x) \right\}$

$\mu_\tau(y) \geq \min \left\{ \mu_\tau((x \rightarrow y) \rightarrow x), \mu_\tau(x) \right\}$ for all $x, y \in A$.

Clearly, $\gamma_\tau(1) \leq \gamma_\tau(x)$ for all $x \in A$.

If $\gamma_\tau(y) = \gamma_\tau((y \rightarrow x) \rightarrow y) \leq \max \left\{ \gamma_\tau((x \rightarrow y) \rightarrow x), \gamma_\tau(x) \right\}$

$\gamma_\tau(y) \leq \max \left\{ \gamma_\tau((x \rightarrow y) \rightarrow x), \gamma_\tau(x) \right\}$ for all $x, y \in A$.

Hence $T = (\mu_\tau, \gamma_\tau)$ is an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra $A$.

IV. CONCLUSION

In this paper, we have introduced the notion of an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra. The properties and equivalent conditions of an intuitionistic fuzzy implicative filter of lattice pseudo-Wajsberg algebra are discussed. Further, we extend this idea as intuitionistic anti-fuzzy implicative filter of lattice pseudo-Wajsberg algebra.

REFERENCES


